CLASSICAL ROOTS OF INTER-UNIVERSAL TEICHMÜLLER THEORY

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"Travel and Lectures"

Overview

Analogy with **étale cohomology, Weil conjectures**

$\longleftrightarrow$ **classical singular (co)homology of topological spaces**

- Isogeny invariance of heights of elliptic curves (Faltings, 1983)
- Crystals and Hodge filtrations (Grothendieck, late 1960's)
- Complex Teichmüller theory (Teichmüller, 1930's)
- Theta function on the upper half-plane (Jacobi, 19-th century)

§1. **Isogeny invariance of heights of elliptic curves**
(cf. [Alien], §2.3, §2.4)

We consider **elliptic curves**.

For l a prime number, the module of ***l*-torsion points** associated to a **Tate curve** $E \stackrel{\mathrm{def}}{=} \mathbb{G}_m/q^{\mathbb{Z}}$ — over, say, $\mathbb{C}$ or a p-adic field — fits into a natural exact sequence:

$$0 \longrightarrow \boldsymbol{\mu}_l \longrightarrow E[l] \longrightarrow \mathbb{Z}/l\mathbb{Z} \longrightarrow 0.$$

Thus, one has **canonical** objects as follows:

a "**multiplicative subspace**" $\boldsymbol{\mu}_l \subseteq E[l]$ and "**generators**" $\pm 1 \in \mathbb{Z}/l\mathbb{Z}$.

$$\mathbb{G}_m \xrightarrow{(\ )^l} \mathbb{G}_m$$

$$q \longmapsto q^l$$

$$\underbrace{\mathbb{G}_m \longrightarrow E}_{\underline{1 \in \mathbb{Z}}}$$

$q \longleftrightarrow$ mult. s/sp. + can. gen.

In the following, we fix an __**elliptic curve**__ E over a __**number field**__ F and a __**prime number**__ $l \geq 5$ such that E has __**stable reduction**__ at all finite places of F.

$\nexists$

GCG

Then, in general, $E[l]$ does __**not**__ admit

GMS a __**global**__ "__**multiplicative subspace**__" and "__**generators**__"

that coincide with the above canonical "multiplicative subspace" and "generators" at __**all finite places**__ where E has __**bad multiplicative reduction**__!

Nevertheless, **suppose** (!!) that such global objects do in fact exist. Then, if we denote by

$$E \to E^*$$

the **isogeny** obtained by forming the **quotient** of E by the

"**global multiplicative subspace**",

then, at each finite prime of bad multiplicative reduction, the respective q-parameters satisfy the following relation:

$$q_E^l = q_{E^*}.$$

If we write $\log(q_E)$, $\log(q_{E^*})$ for the **arithmetic degrees** $\in \mathbb{R}$ determined by these q-parameters, then the above relation takes on the following form:

$$l \cdot \log(q_E) \ = \ \log(q_{E^*}) \ \in \mathbb{R}.$$

On the other hand, if we consider the respective **heights** of the elliptic curves by $\mathrm{ht}_E, \mathrm{ht}_{E^*} \in \mathbb{R}$ — i.e, roughly speaking, **arithmetic degrees** of arithmetic line bundles on F

$$\omega_E^{\otimes 2}, \quad \omega_{E^*}^{\otimes 2}$$

associated to the sheaves of square **differentials** — then we may conclude — cf. the **discriminant mod. form**, regarded as a section of the **ample line bundle** "$\omega_{\overline{\mathcal{M}}_{\mathrm{ell}}}^{\otimes 12}$" on the **compactified moduli stack** $\overline{\mathcal{M}}_{\mathrm{ell}}$ of elliptic curves! — that

$$\mathrm{ht}_{(-)} \approx \frac{1}{6} \cdot \log(q_{(-)})$$

(where "$\approx$" means "up to a discrepancy bounded by a constant").

Moreover, by the famous **computation concerning differentials** due to Faltings (1983), one knows that:

$$\mathrm{ht}_{E^*} \approx \mathrm{ht}_E + \log(l).$$

Thus, (*by ignoring certain subtleties at archimedean places of* F) we conclude that

$$l \cdot \mathrm{ht}_E \lesssim \mathrm{ht}_E + \log(l), \qquad \text{i.e.,} \qquad \mathrm{ht}_E \lesssim \frac{1}{l-1} \cdot \log(l) \lesssim \text{constant}$$

— that is to say, that the height ht_E of the elliptic curve E can be **bounded from above**, and hence (under suitable hypotheses) that there are only **finitely many** isomorphism classes of elliptic curves E that admit a "global multiplicative subspace".

$$\frac{dq}{q} = d\log q \longmapsto \underline{l}\, d\log q$$

Key point:

Consider **distinct elliptic curves** E, E^* such that $q_E^l = q_{E^*}$ (!), but which (up to negligible discrepancies) **share** — i.e., "$\wedge$"! — a **common** $\omega_E \approx \omega_{E^*}$.

One way to understand IUT, esp. Hodge theaters of [IUTchI]:

Apparatus to **generalize** the above argument — by focusing on the above **key point**! — to the case of **general elliptic curves** for which "global multiplicative subspaces", etc. do not necessarily exist.

§2. **Crystals and Hodge filtrations**
(cf. [Alien], §3.1, (iv), (v))

Let X: a smooth, proper, connected *algebraic curve* over $\mathbb{C}$,
$\mathcal{E}$: a *vector bundle* on X.

Consider the *two projections*: $X \xleftarrow{p_1} X \times X \xrightarrow{p_2} X$

Then in general, there exists a vector bundle $\mathcal{F}$ on $X \times X$ such that

$$\left(\mathcal{F} \cong p_1^*\mathcal{E}\right) \quad \vee \quad \left(\mathcal{F} \cong p_2^*\mathcal{E}\right),$$

but there does **not exist** a vector bundle $\mathcal{F}$ on $X \times X$ such that

$$\left(\mathcal{F} \cong p_1^*\mathcal{E}\right) \quad \wedge \quad \left(\mathcal{F} \cong p_2^*\mathcal{E}\right)$$

(which would imply that $\mathcal{E}$ is **trivial**!).

Consider the **first infinitesimal neighborhood** of the **diagonal**

$$X = V(\mathcal{I}) \hookrightarrow X \times X,$$

i.e., $X_{\mathrm{inf}} \stackrel{\mathrm{def}}{=} V(\mathcal{I}^2) \subseteq X \times X$:

> *"moduli space of pairs of points of X (cf. $X \times X$!) that are* **infinitesimally close** *to one another".*

Grothendieck definition of a **connection** on $\mathcal{E}$:

$$p_1^*\mathcal{E}|_{X_{\mathrm{inf}}} \stackrel{\sim}{\to} p_2^*\mathcal{E}|_{X_{\mathrm{inf}}}$$

i.e.,

> *"isomorphism between the fibers of $\mathcal{E}$ at pairs of points of X (cf. $p_1^*\mathcal{E} \stackrel{\sim}{\to} p_2^*\mathcal{E}$ on $X \times X$!) that are* **infinitesimally close** *to one another".*

In general, $\mathcal{E}$ does **not** admit a connection. The **obstruction** to the existence of a connection (cf. Weil!) on $\det(\mathcal{E})$ is a cohomology class in

$$\boxed{H^1(X, \omega_X),}$$

which is in fact equal to the **first Chern class** of $\mathcal{E}$, i.e., from the point of view of de Rham cohomology, the **degree** of $\mathcal{E}$:

$$\boxed{\deg(\mathcal{E}) \in \mathbb{Z}.}$$

Thus, if $\mathcal{E}$ is a *line bundle*, then

$$\mathcal{E} \text{ admits a connection} \quad \iff \quad \deg(\mathcal{E}) = 0.$$

There also exists a *logarithmic version* of this discussion: by considering *logarithmic poles* at a finite number of points of $X(\mathbb{C})$.

H^1_c conns. with nilp. monodromy

Suppose that X is equipped with a *log structure* determined by a finite set of r_X points of $X(\mathbb{C})$. Write $X^{\log}$ for the resulting *log scheme*, $U \subseteq X$ for the *interior* of $X^{\log}$.

Consider a (compactified) **family of elliptic curves**

(non-'cotrivial')

$$f : E \to X$$

(i.e., a family of one-dimensional semi-abelian schemes over X with proper fibers over $U \subseteq X$). Then the **relative first de Rham cohom. module** of this family determines a *rank two vector bundle* on X

$$\boxed{\mathcal{E} \stackrel{\text{def}}{=} \mathbb{R}^1 f_{\mathrm{DR},*}\mathcal{O}_E}$$

equipped with: **Gauss-Manin (logarithmic!) connection** $\nabla_{\mathcal{E}}$ and
a rank one **Hodge subbundle** $\omega_E \subseteq \mathcal{E}$ s.t. $\omega_E \otimes_{\mathcal{O}_X} (\mathcal{E}/\omega_E) \cong \mathcal{O}_X$
(cf. the bundle $\omega_{\overline{\mathcal{M}}_{\mathrm{ell}}}$ of §1!).

Note: ω_E does **not** admit a connection, i.e., in general, $p_1^*\omega_E|_{X_{\mathrm{inf}}}$ is **not isom.** to $p_2^*\omega_E|_{X_{\mathrm{inf}}}$! But one can **measure** the extent to which ω_E **fails** to admit a connection by means of $\nabla_{\mathcal{E}}$, i.e., by considering the (*generically nonzero, $\mathcal{O}_X$-linear!*) composite morphism:

$$\begin{array}{ccc} \omega_E & \hookrightarrow & \mathcal{E} \\ & & \downarrow \nabla_{\mathcal{E}} \\ & & \mathcal{E} \otimes_{\mathcal{O}_X} \omega_X^{\log} \quad \twoheadrightarrow \quad \omega_E^{-1} \otimes_{\mathcal{O}_X} \omega_X^{\log}. \end{array}$$

The resulting **Kodaira-Spencer morphism**

$$\boxed{\kappa_E : \omega_E^{\otimes 2} \hookrightarrow \omega_X^{\log},} \quad \neq 0$$

yields a **bound** ("geometric Szpiro") on the **height** $\deg(\omega_E^{\otimes 2})$ of $f : E \to X$ (cf. §1!):

$$\boxed{\deg(\omega_E^{\otimes 2}) \leq \deg(\omega_X^{\log}) = 2g_X - 2 + r_X.}$$

Key point:

$p_1^*\mathcal{E} \cong p_2^*\mathcal{E}$ serves as a **common** — i.e., "$\wedge$"! — **container** (cf. the *common* "$\omega_E \approx \omega_{E^*}$" of §1!) that is

· **sufficiently large** to house both $p_1^*\omega_E \hookrightarrow p_1^*\mathcal{E}$ and $p_2^*\omega_E \hookrightarrow p_2^*\mathcal{E}$, but
· **sufficiently small** to yield a **nontrivial estimate** on the **height** of the family of elliptic curves $f : E \to X$ under consideration.

One way to understand IUT, esp. multiradial rep. of [IUTchIII]:

Construction — by means of

· **absolute anabelian geometry** and
· the theory of the **étale theta function**

— of a **common container** that is

· **sufficiently large** to house the **incompatible ring structures** on either side of the gluing constituted by the **theta link** $q_E^N \mapsto q_E$, but
· **sufficiently small** to yield **nontrivial estimate** on the **height** of the elliptic curve over a number field under consideration.

§3. **Complex Teichmüller theory**

(cf. [Pano], §2; [Alien], §3.3, (ii))

Relative to a *canonical coordinate* $z = x + iy$ — assoc'd to a *square differential* — on a Riemann surface, **Teichmüller deformations** given by

arith. hol. str.

ring str.

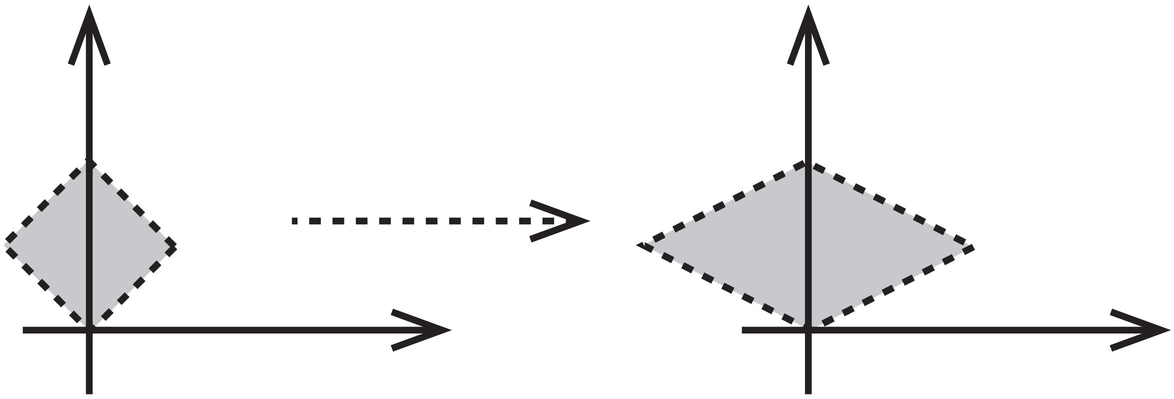
$$\begin{array}{ccccccc} \text{hol. str.} & & & & & & \text{hol. str.} \\ \curvearrowright & & & & & & \curvearrowright \\ \mathbb{C} \ni & z & \mapsto & \zeta & = & \xi + i\eta = \lambda x + iy & \in \mathbb{C} \end{array}$$

— where $1 < \lambda < \infty$ is the **dilation** factor.

Key points:

- **non-hol.** map, but **common real analytic str.** — i.e., "$\wedge$"!
- **one** hol. dim., but **two** underlying real dims., of which **one** is **dilated/deformed**, while the **other** is left **fixed/undeformed**!

Classical complex Teichmüller deformation:



Recall: the **upper half-plane** $\mathfrak{H}$ ($\xrightarrow{\sim}$ **open unit disk $\mathfrak{D}$**) may be regarded as the **moduli space of hol. strs.** on $\mathbb{R}^2$ — cf. the **bijection**:

$$\begin{array}{ccc} \overset{\text{hol. str.}}{\overset{\curvearrowright}{\mathbb{C}^\times}} \backslash GL^+(\mathbb{R}) / \overset{\text{hol. str.}}{\overset{\curvearrowright}{\mathbb{C}^\times}} & \xrightarrow{\sim} & [0,1) \\ \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} & \mapsto & \frac{\lambda-1}{\lambda+1} \end{array}$$

— where

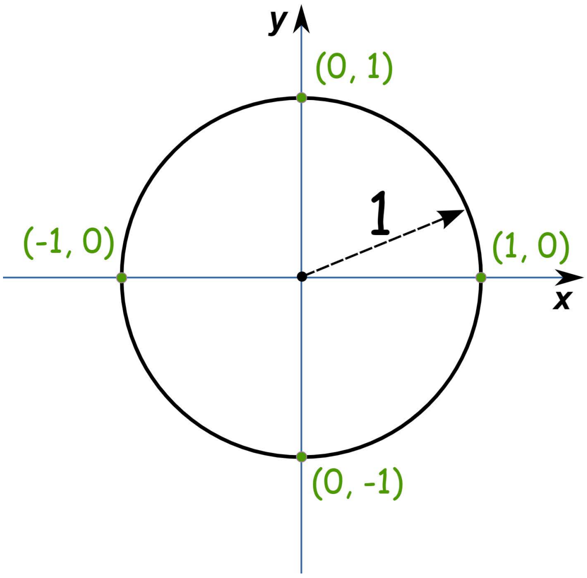
· $\lambda \in \mathbb{R}_{\geq 1}$, and we regard $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ as a **dilation**;

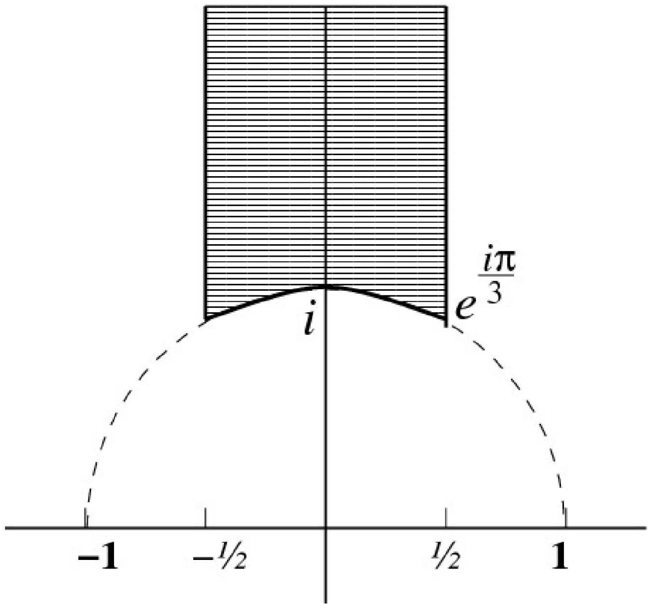
· $GL^+(\mathbb{R})$ denotes the group of 2×2 real matrices with determinant > 0;

$i \in$ · $\mathbb{C}^\times$ denotes the multiplicative group of $\mathbb{C}$, which we regard as a subgroup of $GL^+(\mathbb{R})$ via $a + ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, for $a, b \in \mathbb{R}$ s.t. $(a, b) \neq (0, 0)$.

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ Relative to $GL^+(\mathbb{R}) \curvearrowright \mathfrak{H}$ by linear fractional transformations, $\mathbb{C}^\times$ is the **stabilizer** of $i \in \mathfrak{H}$, so the above **bijection** just states that any $w \in \mathfrak{D}$ may be mapped to $0 \in \mathfrak{D}$ by a **rotation** $\in \mathbb{C}^\times$, followed by a **dilation**.

The **fundamental domain** of the **upper half-plane** and the **unit disk**:
(cf. `https://www.mathsisfun.com/geometry/unit-circle.html` ;
`http://www.math.tifr.res.in/~dprasad/mf2.pdf`)





Key point:

In the discussion of $\mathfrak{H}$: $\mathbb{R}^2$ (with standard orientation) serves as a **common** — i.e., "$\wedge$"! — **container** for various **hol. strs.**

In summary:

$$\begin{array}{ccccc} \textbf{hol. str.} & & & & \textbf{hol. str.} \\ \approx \mathbb{C}^\times & \curvearrowright & \textbf{dilation} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} & \curvearrowleft & \approx \mathbb{C}^\times \end{array}$$

One way to understand IUT, esp. log-theta-lattice of [IUTchIII]:

"infinite H" portion (i.e., portion that is *actually used*) of **log-theta-lattice**:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{array}{ccccc} \textbf{rotations} & & \textbf{dilation} & & \textbf{rotations} \\ \text{of } \textbf{arith.} & & \text{of } \textbf{ring} & & \text{of } \textbf{arith.} \\ \textbf{hol. str.} & & \textbf{str.} \text{ via} & & \textbf{hol. str.} \\ \approx \mathfrak{log} & \curvearrowright & \textbf{theta link} & \curvearrowleft & \approx \mathfrak{log} \end{array}$$

Here, **arith. hol. str.** $\approx$ **ring str.**, which is **not** preserved by **theta link** "$q_E^N \mapsto q_E$"!

The entire **log-theta-lattice** and the **"infinite H"** portion that is *actually used*:

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
 & & \vdots & & \vdots & &
\end{array}
\quad \supseteq \quad
\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\bullet & & \bullet \\
\uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\bullet & \xrightarrow{\Theta} & \bullet \\
\uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\bullet & & \bullet \\
\uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\vdots & & \vdots
\end{array}
$$

§4. **Theta function on the upper half-plane**

(cf. final portion of [Pano], §3; discussion surrounding [Pano], Fig. 4.2)

Recall the **theta function** on $\mathfrak{H} \ni z = x + iy$, where $q \stackrel{\text{def}}{=} e^{2\pi i z}$:

$$\theta(q) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{+\infty} q^{\frac{1}{2}n^2}.$$

Restricting to the **imaginary axis** (i.e., $x = 0$) yields, for $t \stackrel{\text{def}}{=} y$:

$$\theta(t) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 t}.$$

Then the **Jacobi identity** holds:

$$\theta(t) = t^{-\frac{1}{2}} \cdot \theta(t^{-1}).$$

Here, we note that

$$GL^+(\mathbb{R}) \supseteq \mathbb{C}^\times \ni \iota \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

maps $z \mapsto -z^{-1}$, hence $iy \mapsto -iy^{-1}$, i.e., $t \mapsto t^{-1}$.

As one *travels along* the **imag. axis** via $GL^+(\mathbb{R}) \supseteq \mathbb{C}^\times \ni \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) : i \mapsto iy$:

When $|q| \to 0 \iff y \to +\infty$:
$\theta(t)$ series terms are **rapidly decreasing** $\implies$ **easy to compute**!

$$\wedge \quad (!)$$

When $|q| \to 1 \iff y \to +0$:
$\theta(t)$ series terms **not rapidly decreasing** $\implies$ **difficult to compute**!

Note: "$\wedge$" makes sense precisely because one **distinguishes** the ι-*conjugate* regions "$|q| \to 0 \iff y \to +\infty$" and "$|q| \to 1 \iff y \to +0$"!

This situation parallels the **Θ-link** of IUT (cf. $|q^N| \to 0$ vs. $|q| \approx 1$!).

Jacobi identity $\theta(t) \ = t^{-\frac{1}{2}} \cdot \theta(t^{-1})$ may be interpreted as follows:

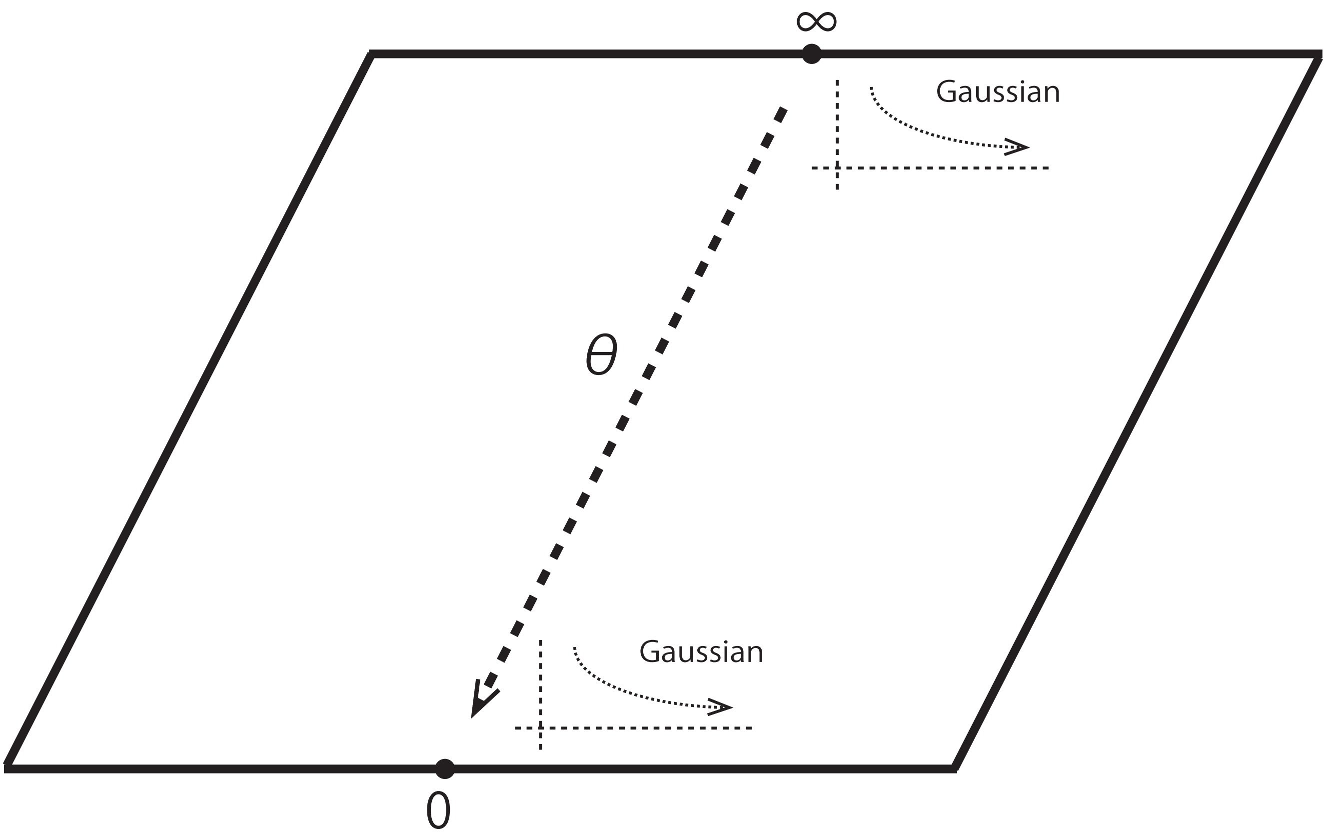
$\theta(t)$ **descends**, *up to a suitable factor* $t^{-\frac{1}{2}}$, to the **quotient** by ι.

Comparison with IUT:

invars. wrt. log-link $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Jacobi identity	$\longleftrightarrow$	**multiradial representation** of IUT
the factor $t^{-\frac{1}{2}}$	$\longleftrightarrow$	**indeterminacies** of multirad. rep.
involution $\iota \in \mathbb{C}^\times$	$\longleftrightarrow$	**log-link** of IUT: rotat. of hol. str.
descent to quotient by ι	$\longleftrightarrow$	**descent** to **single** hol. str./ring str.

Behavior of $\underline{\theta(t)}$ **series terms** upon applying **Jacobi identity**:



Proof of Jacobi identity: One computes $\theta(t^{-1})$ by using the fact that

$$\Big(\text{Fourier transform}\Big)(e^{-t\cdot\square^2}) \approx \underline{\pi^{-\frac{1}{2}}t^{-\frac{1}{2}}} \cdot e^{-\frac{1}{t}\cdot\square^2}$$

— a computation closely related to the computation of the **Gaussian integral**

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \pi^{\frac{1}{2}}$$

via polar coordinates!

This computation is essentially a consequence of the **quadratic form** in the exponent of the **Gaussian**:

$$e^{-t\cdot\text{``}\square^2\text{''}}.$$

quad. form $\approx$ Chern class "$\square^2$" $\Longrightarrow$ **theta group symmetries**

$\Longrightarrow$ **rigidity** properties of étale theta function in IUT

$\Longrightarrow$ **Kummer theory** of **étale theta function** compatible with **$\mathfrak{log}$-link** (cf. "$t \cdot \square^2 \rightsquigarrow \frac{1}{t} \cdot \square^2$" in above computation!) and **multiradial rep.** of IUT

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